Classification of 3-step nilpotent, Einstein Lorentzian Lie algebras with 1-dimensional non-degenerate center

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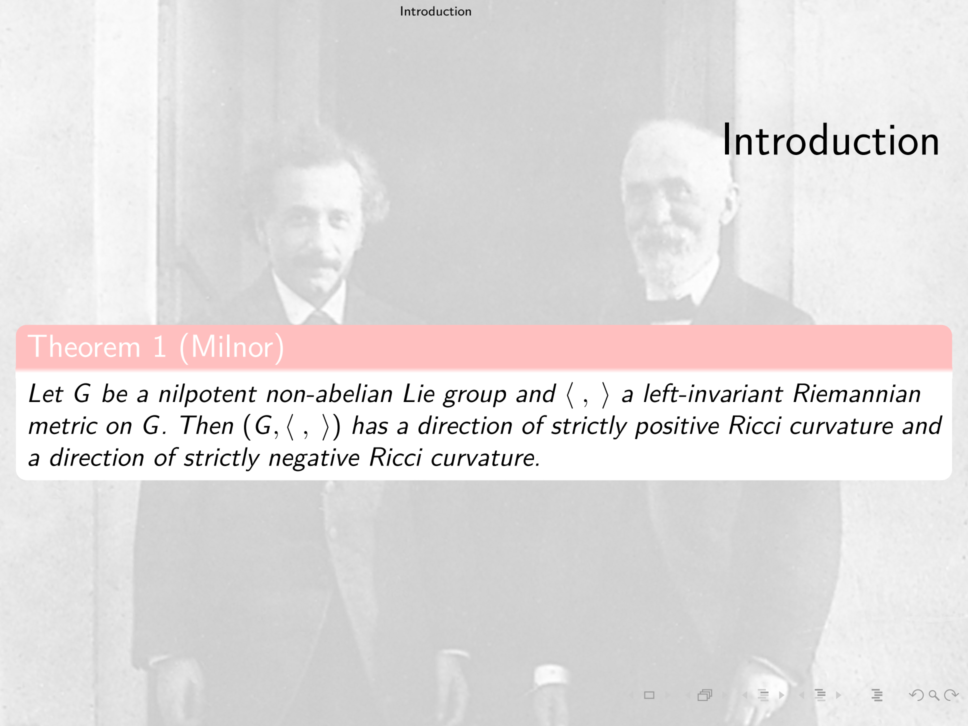
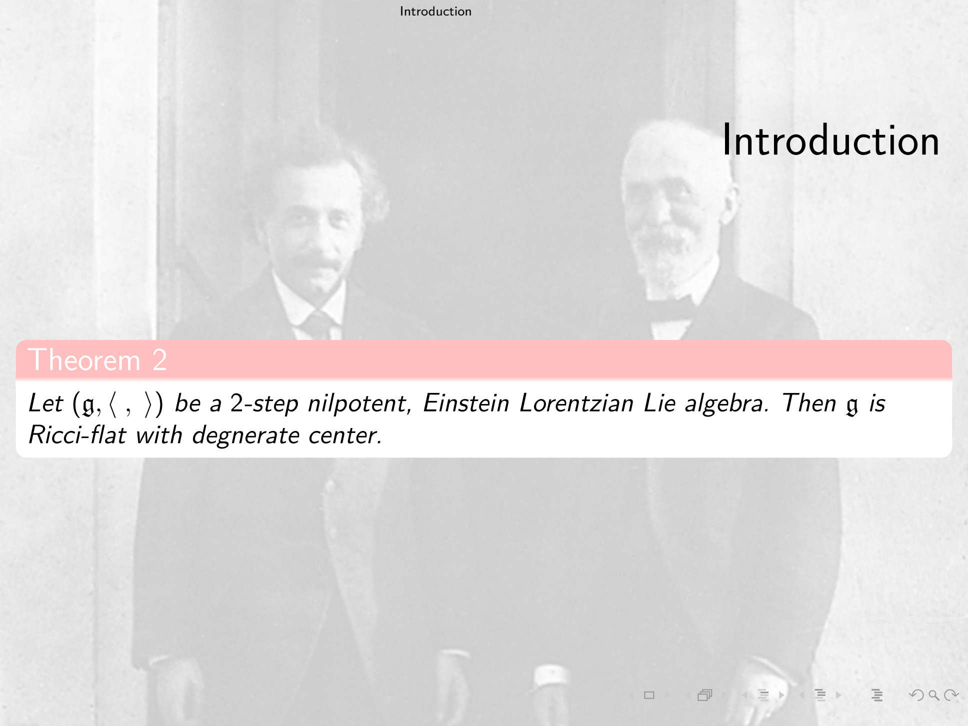


Introduction

Theorem 1 (Milnor)

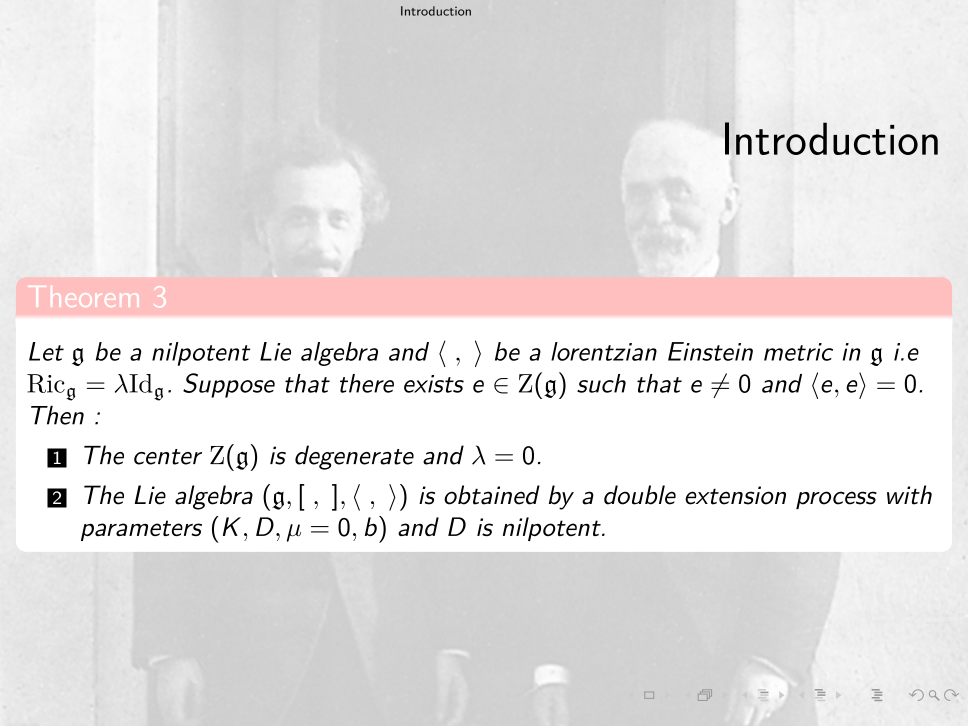
Let G be a nilpotent non-abelian Lie group and $\langle\ ,\ \rangle$ a left-invariant Riemannian metric on G. Then $(G, \langle\ ,\ \rangle)$ has a direction of strictly positive Ricci curvature and a direction of strictly negative Ricci curvature.



Introduction

Theorem 2

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a 2-step nilpotent, Einstein Lorentzian Lie algebra. Then $\mathfrak{g}$ is Ricci-flat with degnerate center.

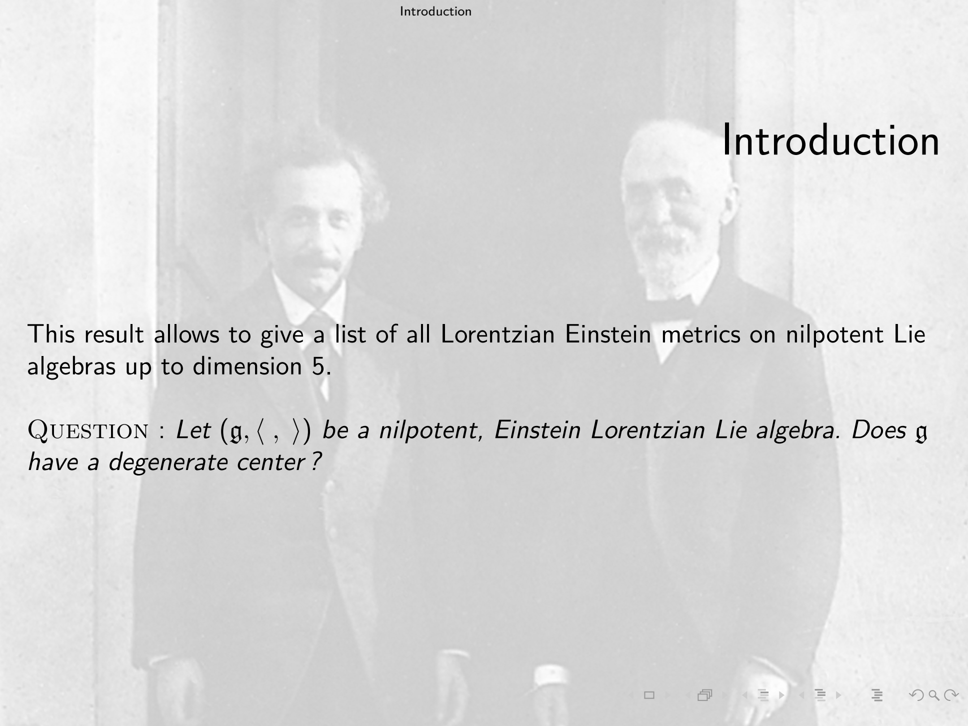


Introduction

Theorem 3

Let $\mathfrak{g}$ *be a nilpotent Lie algebra and* $\langle\ ,\ \rangle$ *be a lorentzian Einstein metric in* $\mathfrak{g}$ *i.e* $\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}}$. *Suppose that there exists* $e \in \mathrm{Z}(\mathfrak{g})$ *such that* $e \neq 0$ *and* $\langle e, e\rangle = 0$. *Then :*

1. *The center* $\mathrm{Z}(\mathfrak{g})$ *is degenerate and* $\lambda = 0$.
2. *The Lie algebra* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *is obtained by a double extension process with parameters* $(K, D, \mu = 0, b)$ *and* D *is nilpotent.*



Introduction

This result allows to give a list of all Lorentzian Einstein metrics on nilpotent Lie algebras up to dimension 5.

QUESTION : *Let* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a nilpotent, Einstein Lorentzian Lie algebra. Does* $\mathfrak{g}$ *have a degenerate center ?*

Introduction

The answer is no. Consider the 6-dimensional Lie algebra $\mathfrak{g} := \mathrm{span}\{e_1, \ldots, e_6\}$ given by :

$$[e_1, e_3] = e_6,\ [e_1, e_5] = e_6,\ [e_2, e_3] = -e_6, [e_2, e_4] = e_6, [e_3, e_4] = e_1, [e_3, e_5] = e_2$$

$$[e_4, e_5] = e_1 + e_2.$$

Then $\mathfrak{g}$ is 3-step nilpotent. Moreover if we define on $\mathfrak{g}$ the inner product $\langle\ ,\ \rangle$ such that $\{e_1, \ldots, e_6\}$ is an orthonormal basis with $\langle e_1, e_1\rangle = -1$, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is Ricci-flat Lorentzian with non-degenerate (Euclidean) center $\mathrm{Z}(\mathfrak{g}) = \mathbb{R}e_6$.

Theorem 4

Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ *be a 3-step nilpotent Lorentzian Lie algebra with 1-dimensional center. Assume* $\mathrm{Z}(\mathfrak{h})$ *is non-degenerate, then* $\langle\ ,\ \rangle$ *is Einstein if and only if it is Ricci-flat and either :*
(i) $\dim \mathfrak{h} = 6$ *and* $\mathfrak{h}$ *is isomorphic to* $\mathrm{L}_{6,19}(-1)$*, i.e.,* $\mathfrak{h}$ *has a basis* $(f_i)_{i=1}^{6}$ *such that the non vanishing Lie brackets are*

$$[f_1, f_2] = f_4, [f_1, f_3] = f_5, [f_2, f_4] = f_6, [f_3, f_5] = -f_6$$

and the metric is given by :

$$\langle\ ,\ \rangle := f_1^* \otimes f_1^* + 2f_2^* \otimes f_2^* + 2f_3^* \otimes f_3^* + 4\alpha^4 f_6^* \otimes f_6^* - 2\alpha^2 f_4^* \odot f_5^*, \quad \alpha \neq 0.$$

(ii) $\dim \mathfrak{h} = 7$ *and* $\mathfrak{h}$ *is an instance of a 1-parameter family of nilpotent Lie algebras given in a basis* $\{f_i\}_{i=1}^{7}$ *by :*

$$[f_1, f_2] = f_5,\ [f_1, f_3] = f_6,\ [f_2, f_3] = f_4,\ [f_6, f_2] = (1-r)f_7,\ [f_5, f_3] = -rf_7,\ [f_4, f_1] = f_7,$$

with $0 < r < 1$*, and the metric has the form :*

$$\langle\ ,\ \rangle = f_1^* \otimes f_1^* + f_2^* \otimes f_2^* + f_3^* \otimes f_3^* - af_4^* \otimes f_4^* + arf_5^* \otimes f_5^* + a(1-r) f_6^* \otimes f_6^* + a^2 f_7^* \otimes f_7^*, a > 0.$$

Preliminaries

Recall that a pseudo-Euclidean vector space is a real vector space $(V, \langle\ ,\ \rangle)$ of finite dimension n endowed with a non degenerate symmetric inner product of signature $(q, p) = (-, \dots, -, +, \dots, +)$. We suppose that $q \leq p$.

1. We say that $(V, \langle\ ,\ \rangle)$ is Euclidean when the signature is equal to $(0, n)$.
2. We say that $(V, \langle\ ,\ \rangle)$ is Lorentzian when the signature is equal to $(1, n-1)$.

Preliminaries

A vector subspace A of V is said to be non-degenerate if the restriction of $\langle\ ,\ \rangle$ is non-degenerate, this the same as $A \cap A^{\perp} = \{0\}$.

For a non-degenerate vector subspace A of a lorentzian vector space V, there are only two situations :

1. A is euclidean and $A^{\perp}$ is lorentzian.
2. A is lorentzian and $A^{\perp}$ is euclidean.

Preliminaries

A vector $u \in V$ is called *spacelike* if $\langle u, u \rangle > 0$, *timelike* if $\langle u, u \rangle < 0$ and *isotropic* if $\langle u, u \rangle = 0$.

A basis $(e_1, \dots, e_n)$ of V is called orthogonal if we have the following property, for any $i, j = 1, \dots, n$ such as $i \neq j$, $\langle e_i, e_j \rangle = 0$. This basis is called orthonormal if it is orthogonal and, for $i = 1, \dots, n$ $\langle e_i, e_i \rangle^2 = 1$.

A *pseudo-Euclidean basis* of V is a basis $(e_1, \bar{e}_2, \dots, e_q, \bar{e}_q, f_1, \dots, f_{n-2q})$ for which the non vanishing products are

$$\langle \bar{e}_i, e_i \rangle = \langle f_j, f_j \rangle = 1, \; i \in \{1, \dots, q\} \quad \text{and} \quad j \in \{1, \dots, n-2q\}.$$

Preliminaries

We will refer to a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a non degenerate symmetric inner product $\langle\ ,\ \rangle$ as a pseudo-Euclidean Lie algebra. The Levi-Civita product is the bilinear map $\mathrm{L} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by the Koszul formula :

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

Its associated curvature is the bilinear map $\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ given by :

$$\mathcal{K}(u, v)w = \mathrm{L}_{[u,v]}w - \mathrm{L}_u \mathrm{L}_v w + \mathrm{L}_v \mathrm{L}_u w,$$

and the Ricci curvature is the bilinear map $\mathrm{ric} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ defined by :

$$\mathrm{ric}(u, v) = \mathrm{tr}(w \mapsto \mathcal{K}(u, w)v).$$

Preliminaries

Denote by $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ the Ricci operator, i.e the endomorphism given by :

$$\mathrm{ric}(u, v) = \langle \mathrm{Ric}(u), v \rangle.$$

Note that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is said to be Einstein if $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}$ with $\lambda \in \mathbb{R}$. It is called Ricci-flat when $\lambda = 0$.

Preliminaries

It is well known that the ricci curvature is also given by :

$$\mathrm{ric}(u,v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v) - \frac{1}{2}\langle \mathrm{ad}_H u, v\rangle - \frac{1}{2}\langle \mathrm{ad}_H v, u\rangle, \quad (1)$$

where J_u is the skew-adjoint endomorphism given by $\mathrm{J}_u v = \mathrm{ad}_v^* u$, and H is the mean curvature vector is defined by the following relation :

$$\langle \mathrm{H}, u\rangle = \mathrm{tr}(\mathrm{ad}_u).$$

Recall that $\mathfrak{g}$ is unimodular if and only if $H = 0$.

Preliminaries

The use of equation (1) above leads to the following interesting formula :

$$\mathrm{Ric} = -\frac{1}{2}(\hat{\mathrm{B}} + \mathcal{J}_1) + \frac{1}{4}\mathcal{J}_2 - \frac{1}{2}(\mathrm{ad}_{\mathrm{H}} + \mathrm{ad}_{\mathrm{H}}^*). \tag{2}$$

The auto-adjoint endomorphism $\hat{\mathrm{B}}$ associated to the Killing form is given by :

$$\langle \hat{\mathrm{B}} u, v \rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v).$$

We denote also by $\mathcal{J}_1$ and $\mathcal{J}_2$ the auto-adjoint endomorphisms defined by :

$$\langle \mathcal{J}_1 u, v \rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) \quad \text{and} \quad \langle \mathcal{J}_2 u, v \rangle = -\mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v) = \mathrm{tr}(\mathrm{J}_u \circ \mathrm{J}_v^*). \tag{3}$$

Preliminaries

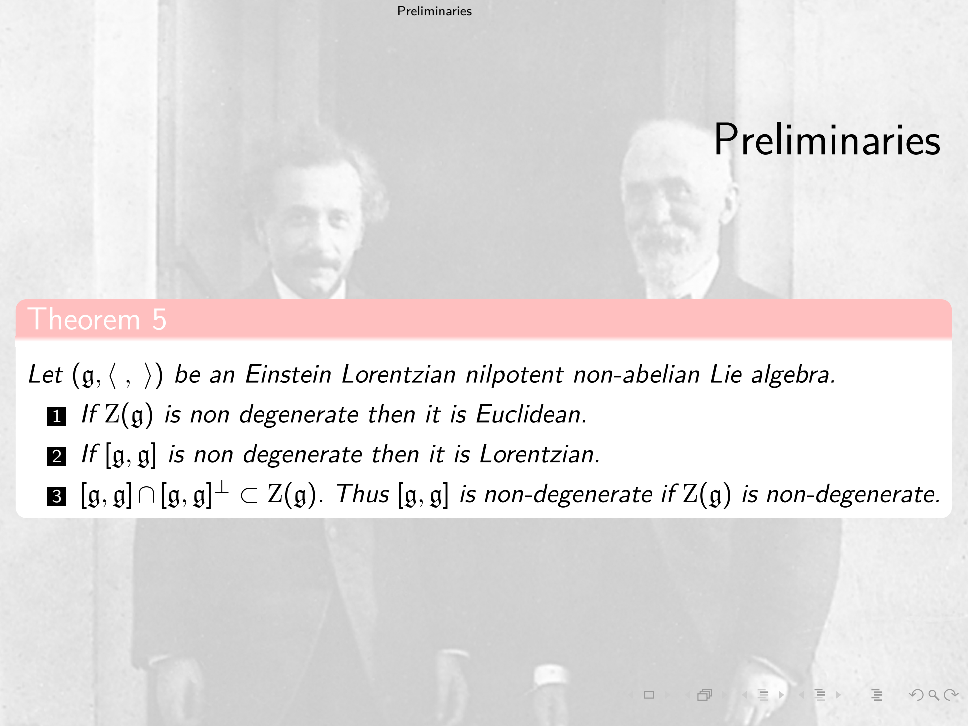
Recall that the Lie algebra $\mathfrak{g}$ is called nilpotent if the lower central series of $\mathfrak{g}$:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}], \ \ k \geq 1,$$

becomes trivial, i.e $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$.

In this case, $\hat{\mathrm{B}} = 0$ and $\mathrm{H} = 0$, therefore the Ricci operator has the following simple expression :

$$\mathrm{Ric} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2. \tag{4}$$



Preliminaries

Theorem 5

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be an Einstein Lorentzian nilpotent non-abelian Lie algebra.*

1. *If* $\mathrm{Z}(\mathfrak{g})$ *is non degenerate then it is Euclidean.*
2. *If* $[\mathfrak{g}, \mathfrak{g}]$ *is non degenerate then it is Lorentzian.*
3. $[\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^{\perp} \subset \mathrm{Z}(\mathfrak{g})$. *Thus* $[\mathfrak{g}, \mathfrak{g}]$ *is non-degenerate if* $\mathrm{Z}(\mathfrak{g})$ *is non-degenerate.*

Main Results

Let $(\mathfrak{h}, \langle\ ,\ \rangle_{\mathfrak{h}})$ be a Lorentzian nilpotent Lie algebra with nondegenerate Euclidean center $Z(\mathfrak{h})$ of dimension $p \geq 1$. Denote by $\langle\ ,\ \rangle_z$ the restriction of $\langle\ ,\ \rangle$ to $Z(\mathfrak{h})$, $\mathfrak{g} = Z(\mathfrak{h})^{\perp}$ and by $\langle\ ,\ \rangle_{\mathfrak{g}}$ the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}$. We get that

$$\mathfrak{h} = \mathfrak{g} \overset{\perp}{\oplus} Z(\mathfrak{h}),$$

where $(Z(\mathfrak{h}), \langle\ ,\ \rangle_z)$ is an Euclidean vector space and $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}})$ is a Lorentzian vector space.

Main Results

For any $u, v \in \mathfrak{g}$, we have the following decomposition :

$$[u, v] = [u, v]_{\mathfrak{g}} + \omega(u, v),$$

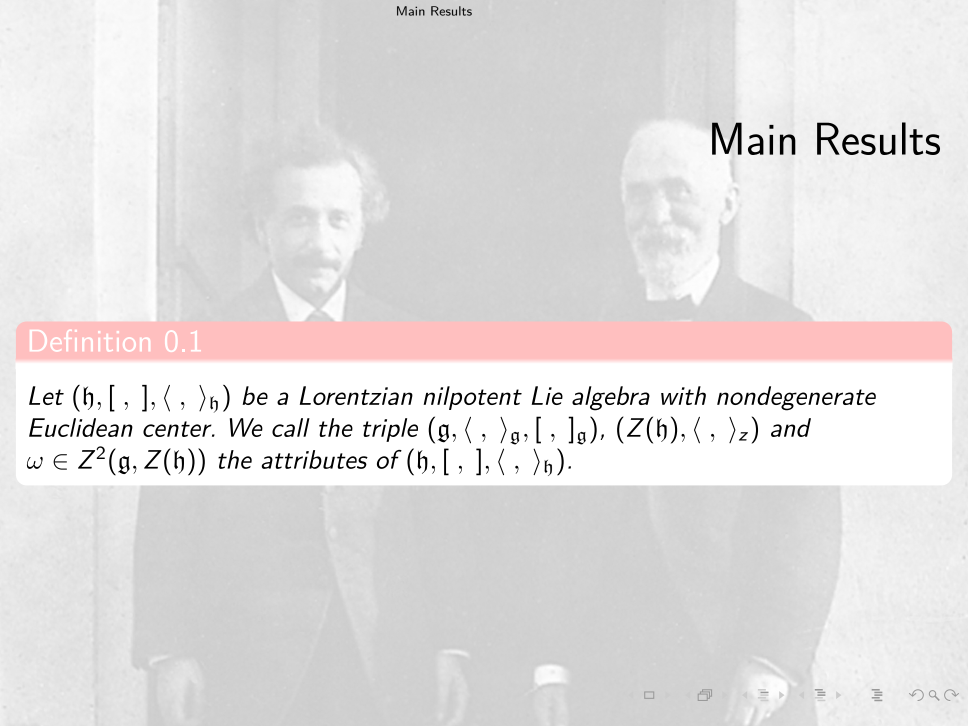
Then $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is a Lie algebra and $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow Z(\mathfrak{h})$ is a 2-cocycle of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ in $Z(\mathfrak{h})$, namely, for any $u, v, w \in \mathfrak{g}$,

$$\omega([u, v]_{\mathfrak{g}}, w) + \omega([v, w]_{\mathfrak{g}}, u) + \omega([w, u]_{\mathfrak{g}}, v) = 0,$$

or simply $d\omega = 0$ where d denotes the Chevalley-Eilenberg differential. Moreover,

$$Z(\mathfrak{g}) \cap \ker \omega = \{0\}, \tag{5}$$

and $(\mathfrak{h}, [\ ,\ ])$ is k-step nilpotent if and only if $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is $(k-1)$-step nilpotent.



Main Results

Definition 0.1

Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$ *be a Lorentzian nilpotent Lie algebra with nondegenerate Euclidean center. We call the triple* $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}}, [\ ,\ ]_{\mathfrak{g}})$*,* $(Z(\mathfrak{h}), \langle\ ,\ \rangle_{z})$ *and* $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$ *the attributes of* $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$*.*

Main Results

We proceed now to express the Ricci curvature of $\mathfrak{h}$ in terms of its attributes

$$(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}}, [\ ,\ ]_{\mathfrak{g}}),\ (Z(\mathfrak{h}), \langle\ ,\ \rangle_{z}) \quad \text{and} \quad \omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h})).$$

For any $x \in Z(\mathfrak{h})$, we define the endomorphism $S_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ by the expression :

$$\langle S_x(u), v\rangle := \langle \omega(u, v), x\rangle.$$

Since ω is alternating, then S_x is skew-symmetric.
Let $\mathcal{B} := (z_1, \ldots, z_p)$ be a basis of $Z(\mathfrak{h})$. There exists a unique family $(S_1, \ldots, S_p)$ of skew-symmetric endomorphisms such that, for any $u, v \in \mathfrak{g}$,

$$\omega(u, v) = \sum_{i=1}^{p} \langle S_i u, v\rangle_{\mathfrak{g}} z_i. \tag{6}$$

This family will be called *ω-structure endomorphisms* associated to $\mathcal{B}$. One can check that $S_i = S_{z_i}$ when $\mathcal{B}$ is orthonormal.

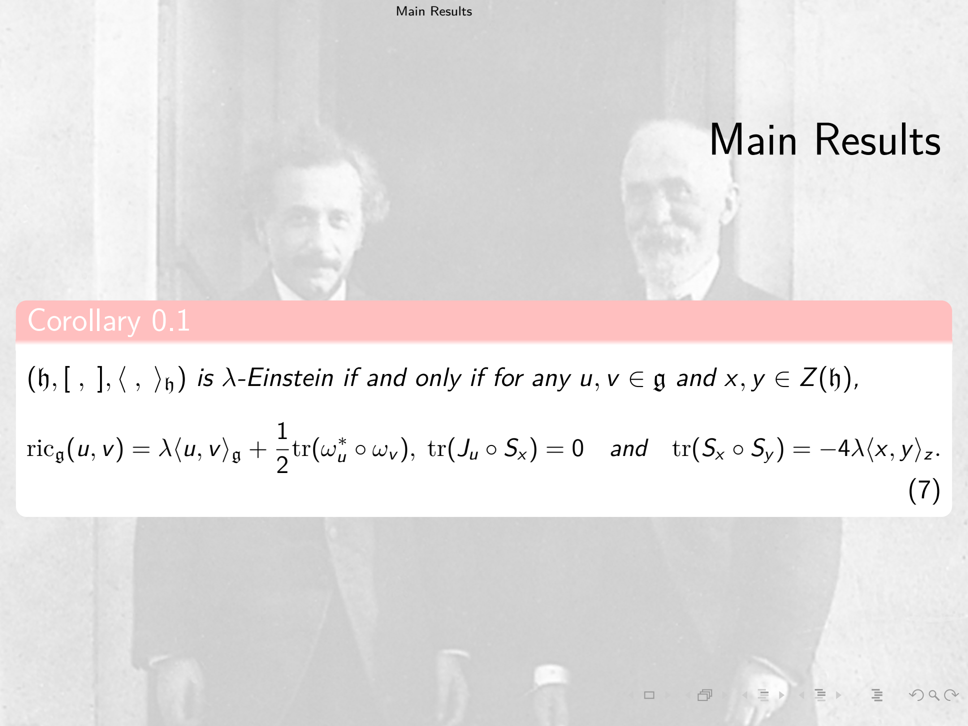
Main Results

Proposition 0.1

The Ricci curvature $\mathrm{ric}_{\mathfrak{h}}$ *of* $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$ *is given by*

$$\begin{aligned}
\mathrm{ric}_{\mathfrak{h}}(u, v) &= \mathrm{ric}_{\mathfrak{g}}(u, v) - \frac{1}{2}\mathrm{tr}(\omega_u^* \circ \omega_v), \quad u, v \in \mathfrak{g},\\
\mathrm{ric}_{\mathfrak{h}}(x, y) &= -\frac{1}{4}\mathrm{tr}(S_x \circ S_y), \quad x, y \in Z(\mathfrak{h}),\\
\mathrm{ric}_{\mathfrak{h}}(u, x) &= -\frac{1}{4}\mathrm{tr}(J_u \circ S_x), \quad x \in Z(\mathfrak{h}), u \in \mathfrak{g},
\end{aligned}$$

where $\mathrm{ric}_{\mathfrak{g}}$ *is the Ricci curvature of* $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$.

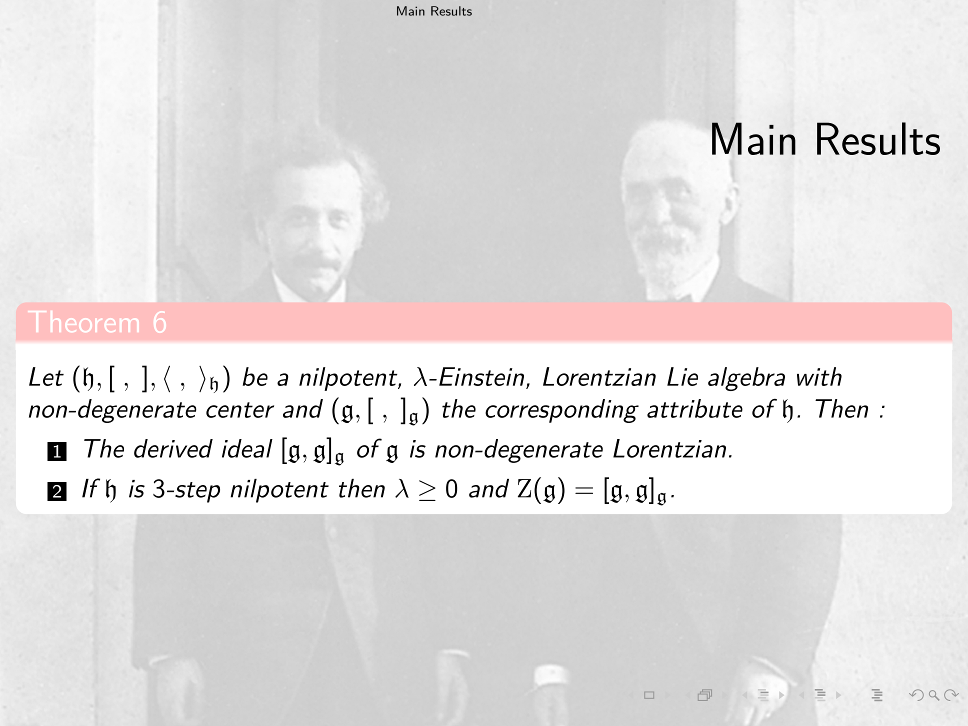


Main Results

Corollary 0.1

$(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$ is λ-Einstein if and only if for any $u, v \in \mathfrak{g}$ and $x, y \in Z(\mathfrak{h})$,

$$\mathrm{ric}_{\mathfrak{g}}(u, v) = \lambda\langle u, v\rangle_{\mathfrak{g}} + \frac{1}{2}\mathrm{tr}(\omega_u^* \circ \omega_v),\ \mathrm{tr}(J_u \circ S_x) = 0 \quad \text{and} \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda\langle x, y\rangle_{\mathfrak{z}}. \tag{7}$$



Main Results

Theorem 6

Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_{\mathfrak{h}})$ *be a nilpotent,* λ*-Einstein, Lorentzian Lie algebra with non-degenerate center and* $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ *the corresponding attribute of* $\mathfrak{h}$*. Then :*

1. *The derived ideal* $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *is non-degenerate Lorentzian.*
2. *If* $\mathfrak{h}$ *is* 3*-step nilpotent then* $\lambda \geq 0$ *and* $\mathrm{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}}$.

Main Results

Definition 0.2

A pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$ *will be called* ω*-quasi Einstein of type* p *if there exists* $\lambda \in \mathbb{R}$ *and a 2-cocycle* ω *with values in a Euclidean vector space* $(V, \langle\ ,\ \rangle_z)$ *of dimension* p *such that* $\ker \omega \cap Z(\mathfrak{g}) = \{0\}$ *and*

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2} D, \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda \langle x, y \rangle_z$$

where $S_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ *denotes the* ω*-structure endomorphism corresponding to* $x \in V$ *and* D *is given by*

$$\langle Du, v \rangle_{\mathfrak{g}} = \mathrm{tr}(\omega_u^* \circ \omega_v)$$

and $\omega_u : \mathfrak{g} \longrightarrow V$, $v \mapsto \omega(u, v)$.

ω-quasi Einstein Lie algebras of type 1

In what follows $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ will always denote a 2-step nilpotent, Lorentzian, ω-quasi Einstein Lie algebra of type 1, with Einstein constant $\lambda \in \mathbb{R}$.

In view of Theorem 6, we get that $\lambda \geq 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Z}(\mathfrak{g})$ is non-degenerate Lorentzian, this allows the decomposition :

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \overset{\perp}{\oplus} [\mathfrak{g}, \mathfrak{g}]^{\perp}. \tag{8}$$

Put $n = \dim[\mathfrak{g}, \mathfrak{g}]$ and $m = \dim[\mathfrak{g}, \mathfrak{g}]^{\perp}$. The crucial part is to adapt the quasi-Einstein system on $\mathfrak{g}$

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2} D, \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda \langle x, y \rangle_z,$$

to the decomposition given by (8), this leads to the following system :

$$
\begin{cases}
-\frac{1}{2}J_1^2 + \frac{1}{2}\sum_{j=2}^{n} J_j^2 + \frac{1}{2}(L^2 - BB^*) = \lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]^\perp}, \\
-2B^*B - \sum_{i,j=1}^{n} \langle e_i, u\rangle \mathrm{tr}(J_i \circ J_j) e_j = 4\lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]}, \\
\mathrm{tr}(L^2) - 2\mathrm{tr}(BB^*) = -4\lambda, \\
LB = 0.
\end{cases}
\tag{9}
$$

This system is obtained by observing that $D = -S^2$ and then writing :

$$
S = \begin{pmatrix} 0 & -B^* \\ B & L \end{pmatrix},\ D = \begin{pmatrix} B^*B & B^*L \\ -LB & BB^* - L^2 \end{pmatrix},\ \mathrm{Ric}_{\mathfrak{g}} = \begin{pmatrix} \frac{1}{4}\mathcal{J}_2 & 0 \\ 0 & -\frac{1}{2}\mathcal{J}_1 \end{pmatrix}
$$

It then suffices to express $\mathcal{J}_1$ and $\mathcal{J}_2$ by means of structure endomorphisms $J_1, \ldots, J_n$ of $\mathfrak{g}$ corresponding to some orthonormal basis $\{e_1, \ldots, e_n\}$ of $[\mathfrak{g},\mathfrak{g}]$ with $\langle e_1, e_1\rangle = -1$.

We can in fact show that there exists an orthonormal basis $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ of $[\mathfrak{g}, \mathfrak{g}]$ with $\langle e_1, e_1 \rangle = -1$ and an orthonormal basis $\mathcal{B}_2 = \{f_1, \ldots, f_m\}$ of $[\mathfrak{g}, \mathfrak{g}]^\perp$ such that system (9) is given in $\mathcal{B}_1 \cup \mathcal{B}_2$ as follows :

$$\begin{cases} B^*B = \mathrm{Diag}(-(2\lambda + \frac{1}{2}\langle e_i, e_i\rangle \mathrm{tr}(J_i^2)), \ 1 \leq i \leq n) \\ \mathrm{tr}(J_i \circ J_j) = 0, \ i \neq j \\ L^2 = \mathrm{Diag}(0, \ldots, 0, -\mu_1^2, -\mu_1^2, \ldots, -\mu_r^2, -\mu_r^2), \ \mu_i \in \mathbb{R}, \end{cases} \tag{10}$$

and

$$J_1^2 - \sum_{k=2}^{n} J_k^2 = \mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(J_1^2), \frac{1}{2}\mathrm{tr}(J_2^2), \ldots, \frac{1}{2}\mathrm{tr}(J_n^2), -2\lambda, \ldots, -2\lambda, -(2\lambda + \mu_1^2), \ldots, -(2\lambda + \mu_r^2)\right). \tag{11}$$

The first result concerning this system of equations is a direct consequence of the following Lemma :

Lemma 7

Let $M_1, \dots, M_n$ be a family of skew-symmetric $m \times m$ matrices with $2 \leq n \leq m$ and let $(v_1, \dots, v_{m-n})$ be a family of nonpositive real numbers such that :

$$M_1^2 - \sum_{l=2}^{n} M_l^2 = \mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(M_1^2), \frac{1}{2}\mathrm{tr}(M_2^2), \dots, \frac{1}{2}\mathrm{tr}(M_n^2), v_1, \dots, v_{m-n}\right). \quad (12)$$

Then

$$(v_1, \dots, v_{m-n}) = (0, \dots, 0).$$

Moreover, for any $i \in \{2, \dots, n\}$, $\mathrm{rank}(M_i) \leq 2$. If furthermore $\mathrm{tr}(M_i \circ M_j) = 0$, $i \neq j$ then we can find an orthonomal basis $\{u_1, \dots, u_n, w_1, \dots, w_{m-n}\}$ of $\mathbb{R}^m$ such that for all $2 \leq i, j \leq n$:

$$M_i(u_1) = \alpha_i u_i, \; M_i(u_j) = -\delta_{ij}\alpha_i u_1 \quad \textit{and} \quad M_i(w_l) = 0.$$

ω-quasi Einstein Lie algebras of type 1

Applying this lemma to our situation, we get $\lambda = 0$, i.e $\mathfrak{g}$ is quasi-Ricci-flat, $L = 0$ and that there exists an orthonormal basis $\{u_1, \ldots, u_n, v_1, \ldots, v_{m-n}\}$ of $[\mathfrak{g}, \mathfrak{g}]^\perp$ such that for all $2 \leq i, j \leq n$:

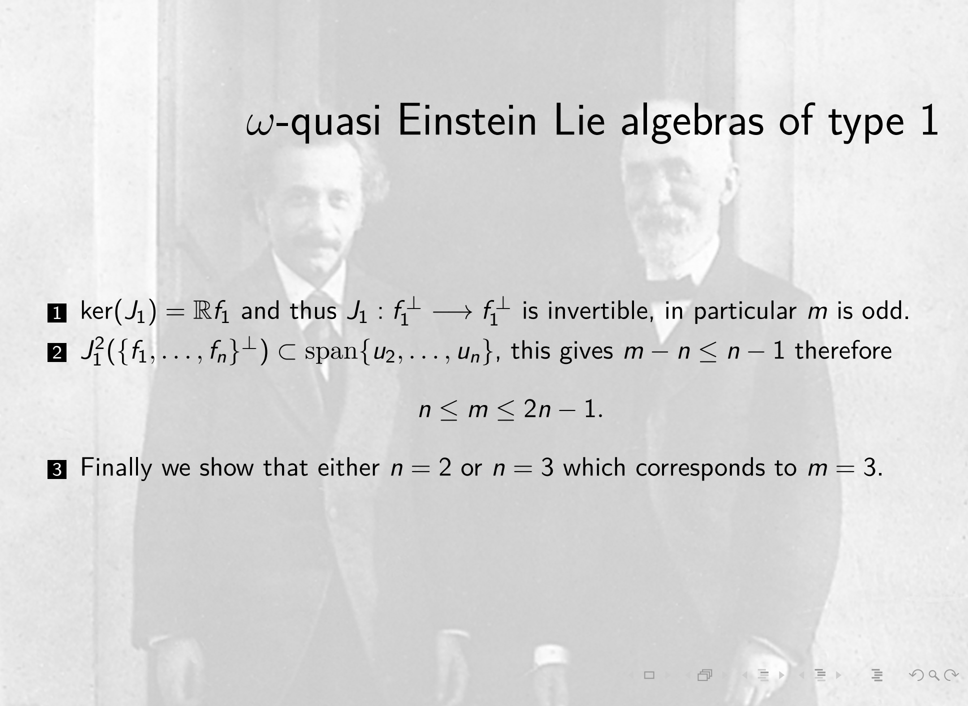
$$J_i(u_1) = \alpha_i u_i,\ J_i(u_j) = -\delta_{ij}\alpha_i u_1, \quad J_i(v_k) = 0.$$

The previous system is then reduced to the following set of equations :

$$J_1^2(f_1) = J(f_1) + \alpha^2 f_1,\ J_1^2(f_i) = J(f_i) - \alpha_i^2 f_i \quad \text{and} \quad J_1^2(v) - J(v) = 0. \tag{13}$$

$$B^*B(e_1) = -\alpha^2 e_1, \ \ B^*B(e_i) = \alpha_i^2 e_i. \tag{14}$$

for $i = 2, \ldots, n$, $v \in \{f_1, \ldots, f_n\}^\perp$, $\alpha^2 = \alpha_2^2 + \cdots + \alpha_n^2$ and $J = J_2^2 + \cdots + J_n^2$.



ω-quasi Einstein Lie algebras of type 1

1. $\ker(J_1) = \mathbb{R}f_1$ and thus $J_1 : f_1^{\perp} \longrightarrow f_1^{\perp}$ is invertible, in particular m is odd.
2. $J_1^2(\{f_1, \ldots, f_n\}^{\perp}) \subset \mathrm{span}\{u_2, \ldots, u_n\}$, this gives $m - n \leq n - 1$ therefore

$$n \leq m \leq 2n - 1.$$

3. Finally we show that either $n = 2$ or $n = 3$ which corresponds to $m = 3$.

1. If $n = 2,\ m = 3$ i.e $\dim \mathfrak{g} = 5$, we get that $B(e_1) = au_1,\ B(e_2) = bv_1$, and the structure endomorphisms $J_1,\ J_2$ of $\mathfrak{g}$ are represented in the basis $\{u_1, u_2, v_1\}$ of $[\mathfrak{g}, \mathfrak{g}]^\perp$ by the matrices

$$J_2 = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{pmatrix} \quad \text{and} \quad a^2 = b^2 = \alpha^2.$$

2. If $n = m = 3$ i.e $\dim \mathfrak{g} = 6$, then we find that $B(e_2) = \mp\epsilon a u_3,\ B(e_3) = \pm\epsilon b u_2$ and $B(e_1) = \pm c u_1$, moreover J_1, J_2, J_3 are represented in the basis $\{u_1, u_2, u_3\}$ by the matrices :

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon c \\ 0 & \epsilon c & 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ J_3 = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & O \\ b & 0 & 0 \end{pmatrix},$$

with $c = \sqrt{a^2 + b^2}$.

Theorem 8

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *be a Lorentzian 2-step nilpotent Lie algebra and* $\omega \in \mathrm{Z}^2(\mathfrak{g}, \mathbb{R})$*. Then* $\mathfrak{g}$ *is* ω*-quasi* λ*-Einstein of type* 1 *if and only if* $\lambda = 0$ *and, up to an isomorphism,* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle, \omega)$ *has one of the following forms :*

1. $\dim \mathfrak{g} = 5$ *and there exists an orthonormal basis* $\{e_1, e_2, u_1, u_2, u_3\}$ *of* $\mathfrak{g}$ *with* $\langle e_1, e_1\rangle = -1$ *such that the non vanishing Lie brackets and* ω*-products are given by :*

$$[u_1, u_2] = \alpha e_2,\ [u_2, u_3] = \pm\alpha e_1,\ \omega(e_2, u_3) = \epsilon\alpha,\ \omega(e_1, u_1) = \mp\epsilon\alpha, \quad \alpha \neq 0, \epsilon = \pm 1. \tag{15}$$

2. $\dim \mathfrak{g} = 6$ *and there exists an orthonormal basis* $\{e_1, e_2, e_3, u_1, u_2, u_3\}$ *of* $\mathfrak{g}$ *such that* $\langle e_1, e_1\rangle = -1$ *and the non vanishing Lie brackets and* ω*-products are given by :*

$$\begin{cases} [u_1, u_2] = \alpha_2 e_2,\ [u_1, u_3] = \alpha_3 e_3,\ [u_2, u_3] = \epsilon\alpha e_1, \\ \omega(e_2, u_3) = \mp\epsilon\alpha_2,\ \omega(e_3, u_2) = \pm\epsilon\alpha_3,\ \omega(e_1, u_1) = \pm\alpha, \end{cases} \tag{16}$$

where $\alpha_2, \alpha_3 \neq 0$*,* $\epsilon = \pm 1$ *and* $\alpha = \sqrt{\alpha_2^2 + \alpha_3^2}$*.*

A Lorentzian 3-step nilpotent Lie algebras $(\mathfrak{h}, \langle\ ,\ \rangle)$ with non-degenerate 1-dimensional center is Einstein if and only if it is Ricci-flat and has one of the following forms :

1. Either $\dim \mathfrak{h} = 6$ in which case $\dim[\mathfrak{h}, \mathfrak{h}] = \mathrm{codim}[\mathfrak{h}, \mathfrak{h}] = 3$ and there exists an orthonormal basis $\{x, e_1, e_2, u_1, u_2, u_3\}$ of $\mathfrak{h}$ with $\langle e_1, e_1\rangle = -1$ such that the Lie algebra structure is given by :

$$[u_1, u_2] = \alpha e_2,\ [u_2, u_3] = \pm\alpha e_1,\ [e_2, u_3] = \alpha x,\ [e_1, u_1] = \mp\alpha x, \quad \alpha \neq 0. \tag{17}$$

   or

$$[u_1, u_2] = \alpha e_2,\ [u_2, u_3] = \pm\alpha e_1,\ [e_2, u_3] = -\alpha x,\ [e_1, u_1] = \pm\alpha x, \quad \alpha \neq 0. \tag{18}$$

2. $\dim \mathfrak{h} = 7$ in which case $\dim[\mathfrak{h}, \mathfrak{h}] = \mathrm{codim}[\mathfrak{h}, \mathfrak{h}] + 1 = 4$. Moreover there exists an orthonormal basis $\{x, e_1, e_2, e_3, u_1, u_2, u_3\}$ of $\mathfrak{h}$ such that $\langle e_1, e_1\rangle = -1$ and in which the Lie algebra structure is given by :

$$\begin{gathered}[u_1, u_2] = \alpha_2 e_2,\ [u_1, u_3] = \alpha_3 e_3,\ [u_2, u_3] = \epsilon\alpha e_1,\ [e_2, u_3] = \mp\epsilon\alpha_2 x, \\ [e_3, u_2] = \pm\epsilon\alpha_3 x,\ [e_1, u_1] = \pm\alpha x\end{gathered} \tag{19}$$

   where $\alpha = \sqrt{\alpha_2^2 + \alpha_3^2}$.

END

Thanks for your attention